Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Applied Mathematics and Mechanics 70 (2006) 56–64

Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

An inverted pendulum on a fixed and a moving base ☆

A.M. Formal'skii

Moscow, Russia

Received 14 April 2005

Abstract

The plane motions of a controlled single-link pendulum with a fixed suspension point and a pendulum with its suspension point located at the centre of a wheel which rolls without sliding along a flat horizontal surface are considered. The control torque, applied to the pendulum at the suspension point, is bounded in absolute magnitude. A controllability domain is constructed in the linear approximation for the one and the other pendulum, from all points of which the pendulum can be brought into the upper unstable equilibrium position without oscillations about the lower equilibrium. It is shown that the domain of controllability is greater for a pendulum mounted on a wheel, as a result it is more easily stabilizable. Control laws are constructed, under which the domain of attraction is identical to the controllability domain and is thereby the largest possible domain.

Many investigations (see Ref. 1–7, for example) have been devoted to problems of controlling a single-link pendulum and its stabilization in the upper unstable equilibrium position. Such problems are not only of theoretical interest but also of practical interest, and possible applications touch on problems of the control of a monocycle,[8,9,a] for example.

The constraints imposed on the control torque, which is applied to the pendulum at the suspension point, can be hard. In this case, bringing the pendulum into the upper unstable equilibrium position without oscillations about a lower equilibrium is not possible from all initial states. By linearizing the equations of motion of the pendulum about the upper unstable equilibrium position, it is possible to construct (analytically) the set of states, from which this equilibrium can be achieved.

1. A pendulum with a fixed suspension point

A single-link pendulum, to which an external torque L is applied at the fixed suspension point O, is shown in Fig. 1; β is the angle of deflection of the pendulum from the vertical measured counterclockwise. The torque L is assumed to be positive and is directed counterclockwise.

The equation of motion of such a pendulum is well known:

$$mr^2\ddot{\beta} - mgb\sin\beta = L \tag{1.1}$$

Here, m is the mass of the pendulum, b is the distance from the suspension point O to the centre of mass of the pendulum, r is the radius of inertia of the pendulum about the suspension point O and g is the acceleration due to gravity. The friction force in the axis of suspension is ignored.

☆ *Prikl. Mat. Mekh.* Vol. 70, No. 1, pp. 62–71, 2006.
E-mail address: formal@imec.msu.ru.
[a] See also: www.theriotwheel.com, www.jackiechabanias.com, www.americanroadshop.com, www.dself.dsl.pipex.com, and www.segway.com

0021-8928/$ – see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.jappmathmech.2006.03.010

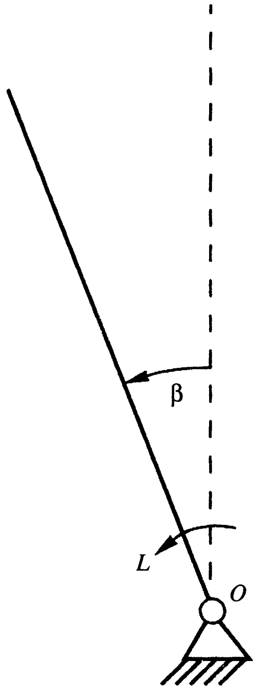

Fig. 1.

Introducing the dimensionless time τ and the dimensionless torque μ by the formulae

$$t = T\tau \left(T^2 = \frac{r^2}{gb}\right), \quad \mu = \frac{L}{mgb} \tag{1.2}$$

we rewrite Eq. (1.1) in the form

$$\beta'' - \sin\beta = \mu \tag{1.3}$$

A prime denotes differentiation with respect to the dimensionless time τ.

We shall assume that the torque L, which is developed using an electric motor, for example, is bounded in absolute magnitude:

$$|L(t)| \le L_0, \text{ where } L_0 = \text{const}, \text{ or } |\mu| \le \mu_0, \text{ where } \mu_0 = \frac{L_0}{mgb} \tag{1.4}$$

We shall call the piecewise-continuous functions $\mu(t)$, which satisfy inequality (1.4), permissible controls and we shall denote the set of permissible controls by W.

If the constant L_0 is sufficiently large, that is, $L_0 > mgb$ ($\mu_0 > 1$), the pendulum can be brought from any initial state

$$\beta(0), \quad \beta'(0) = 0 \tag{1.5}$$

using the control $\mu(t) \in W$, into the upper unstable equilibrium position

$$\beta = 0, \quad \beta' = 0 \ (\dot{\beta} = 0) \tag{1.6}$$

by monotonically changing the angle β. There is no difficulty in constructing the feedback $\mu(\beta, \beta')$, which ensures the stabilization (asymptotic stability) of the equilibrium position (1.6), that is, which brings the pendulum into equilibrium from initial states located within a certain neighbourhood of this equilibrium position. The feedback can be linear with saturation (see below).

We shall assume that a "hard" constraint is imposed on the torque L that is, $L_0 < mgb$ ($\mu_0 < 1$), then, not from all states (1.5) it is possible to bring the pendulum into the equilibrium (1.6) without oscillations about the lower equilibrium position

$$\beta = \pi, \quad \beta' = 0 \tag{1.7}$$

From some of its states, it can only be brought into the equilibrium (1.6) using a bang-bang control after a certain number of oscillations (with increasing amplitude) about the position (1.7).[5,6] It is possible to bring the pendulum into the equilibrium (1.6) from a number of other states with a single transition through the position $\beta = \pi$.

We will now consider the problem of stabilizing (maintaining) the pendulum in the upper unstable equilibrium position, assuming that, at the beginning of the stabilization process, it is already in a certain fairly small neighbourhood of this desired position. Hence, we exclude from the consideration the circular motions of the pendulum, that is, motions, during which it passes through the position $\beta = \pi$.

We linearize Eq. (1.3) about the state (1.6) assuming that the pendulum only slightly deviates from the vertical during the motion. Then, instead of (1.3), we obtain the linear equation

$$\beta'' - \beta = \mu \tag{1.8}$$

When there is no control, that is, when $\mu = 0$, one eigenvalue of Eq. (1.8) is positive and the other is negative.

We now reduce Eq. (1.8) to a system of two first-order differential equations in the canonical variables

$$y' = y + \mu; \quad y = \beta + \beta' \tag{1.9}$$

$$z' = -z - \mu; \quad z = \beta - \beta' \tag{1.10}$$

The differential equation (1.9) describes the behaviour of the "unstable "variable y, corresponding to the positive eigenvalue, while Eq. (1.10) describes the behaviour of the "stable "variable z.

The set of initial states, for each of which a control $\mu(t) \in W$ exists such that the solution of Eq. (1.8) (of system (1.9), (1.10)) with this control comes to the equilibrium (1.6), is called the controllability domain.[10] This domain, which we shall denote by Q, is described[10] by the inequality

$$|y| < \mu_0 \quad (|\beta + \beta'| < \mu_0) \tag{1.11}$$

that is, the set Q of initial states from which the system can be brought into the equilibrium (1.6) is only bounded with respect to the unstable coordinate y. Inequality (1.11) describes a strip in the plane of the variables β, β', the boundaries of which are two parallel straight lines at the same distance from the origin of coordinates. This strip Q is shown in Fig. 2.

If the initial velocity $\beta'(0) = 0$, then the constraint on the initial angle, at which the pendulum can be brought into equilibrium (1.6) has the form

$$|\beta(0)| < \mu_0 \tag{1.12}$$

Note that under the limiting control actions $\mu(t) = \mp\mu_0$ the states

$$\beta = \pm\mu_0, \quad \beta' = 0 \tag{1.13}$$

are equilibrium states, and they lie on the boundary of the controllability domain. The states

$$\beta = \pm\arcsin\mu_0, \quad \beta' = 0$$

will be equilibrium states for non-linear equation (1.3) when $\mu(t) = \mp\mu_0$.

If the initial angle $\beta(0) = 0$, the constraint on the initial velocity, at which it is possible to bring the pendulum into equilibrium (1.6), has the form

$$|\beta'(0)| < \mu_0 \tag{1.14}$$

With the initial conditions $\beta(0) = 0$, $\beta'(0) = \pm\mu_0$ and controls $\mu(t) = \mp\mu_0$, the solutions of Eq. (1.8) tend asymptotically to the points (1.13). The corresponding phase trajectories are segments of the straight lines bounding the controllability domain (1.11).

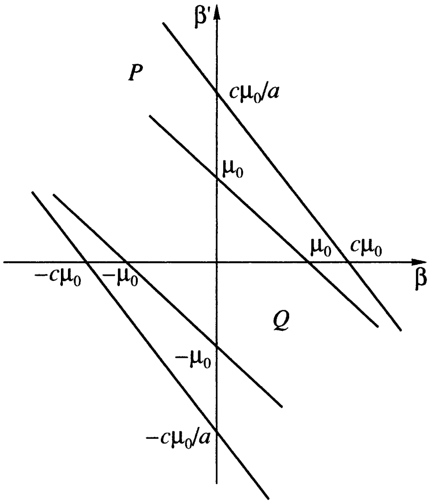

Fig. 2.

It can be shown that the asymptotic stability of the solution (1.6) of Eq. (1.8) (of system (1.9), (1.10)) holds under a linear feedback

$$\mu = \gamma y = \gamma(\beta + \beta'), \quad \gamma < -1$$

and, so also under a linear feedback with saturation

$$\mu = \begin{cases} -\mu_0 & \text{when} \quad \gamma(\beta + \beta') \le -\mu_0 \\ \gamma(\beta + \beta') & \text{when} \quad |\gamma(\beta + \beta')| \le \mu_0 \\ \mu_0 & \text{when} \quad \gamma(\beta + \beta') \ge \mu_0 \end{cases} \tag{1.15}$$

It follows from known results[4–6] that, under feedback (1.15), the domain of attraction of the desired equilibrium position (1.6) is identical to the controllability domain Q (1.11), that is, feedback (1.15) gives the greatest possible domain of attraction (in the linear approximation) and is, in this sense, optimal. Actually, if $y(0) \in Q$, then, under control (1.15), the solution of Eq. (1.9), $y(t) \to 0$ as $t \to \infty$. If $y(t) \to 0$, then the function $\mu(t)$, which is defined by expression (1.15), also tends to zero, $\mu(t) \to 0$. Then, as $t \to \infty$, the solution of Eq. (1.10) $z(t) \to 0$ for any initial value $z(0)$.

Note that the problem of stabilizing the magnetic suspension in a gradiometer has been solved[11] by "suppressing" instability with respect to a single canonical variable.

2. A pendulum with a moving suspension point

We will now consider a pendulum, the suspension point of which is situated at the centre O of a wheel (Fig. 3). The wheel, which is symmetrical about its axis O, can roll without sliding over a flat horizontal surface in a straight line. We will denote the mass of the wheel by M, its radius by R and the radius of inertia about the centre O by ρ. We will denote the angle of counterclockwise rotation of some fixed radius (marked on the wheel), which, at the start of the motion, is directed along the horizontal axis X, by φ and we will denote by x the displacement of the centre of mass O along a horizontal straight line such that $\dot{x} = -\dot{\varphi}R$. As above, suppose β is the angle of deflection of the pendulum from the vertical, m is its mass, b is the distance from the suspension point O to its centre of mass and r is the radius of inertia of the pendulum about the suspension point O. We shall assume that an electric motor is mounted on the axis of the wheel with its stator rigidly fastened to the wheel and its rotor rigidly fastened to the pendulum. Suppose torque L, developed by this motor, tend to rotate the pendulum counterclockwise and simultaneously tends to rotate the wheel clockwise. Hence, L is an internal torque.

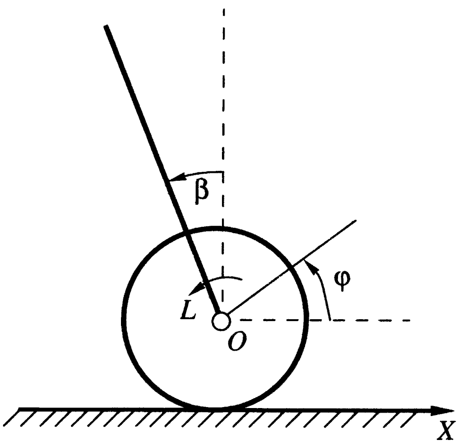

Fig. 3.

The kinetic energy of this system of two bodies has the form

$$E = \frac{1}{2}(a_{11}\dot{\varphi}^2 + 2a_{12}\cos\beta\dot{\varphi}\dot{\beta} + a_{22}\dot{\beta}^2) \tag{2.1}$$

where

$$a_{11} = M(R^2+\rho^2)+mR^2, \quad a_{12} = mRb, \quad a_{22} = mr^2 \tag{2.2}$$

All the coefficients in (2.2) are positive.

The potential energy and the virtual work have the form

$$\Pi = mgb\cos\beta, \quad \delta W = L(\delta\beta - \delta\varphi) \tag{2.3}$$

By Lagrange's method of the second kind, we derive the equations of motion of the system

$$a_{11}\dot{\omega} + a_{12}\cos\beta\ddot{\beta} - a_{12}\sin\beta\dot{\beta}^2 = -L, \quad a_{12}\cos\beta\dot{\omega} + a_{22}\ddot{\beta} - mgb\sin\beta = L \tag{2.4}$$

using expressions (2.1) and (2.3). Here, $\omega = \dot{\varphi}$ is the angular velocity of the wheel. The mechanical system being considered has two degrees of freedom. However, the angle of rotation of the wheel φ is a cyclic variable, and the system of equations (2.4), the order of which is equal to three, can be used to describe the motion. By integrating the equation $\dot{x} = -\omega R$, we can find the change in the coordinate x of the wheel. Note that, on adding Eqs. (2.4), we obtain an equation, which describes the change in the angular momentum about the point of contact of the wheel with the surface.

The equations can to be represented in the form

$$(a_{11}a_{22} - a_{12}^2\cos^2\beta)\ddot{\beta} + a_{12}^2\dot{\beta}^2\sin\beta\cos\beta - a_{11}mgb\sin\beta = (a_{11} + a_{12}\cos\beta)L \tag{2.5}$$

$$(a_{11}a_{22} - a_{12}^2\cos^2\beta)\dot{\omega} - a_{12}a_{22}\dot{\beta}^2\sin\beta + a_{12}mgb\sin\beta\cos\beta = -(a_{22} + a_{12}\cos\beta)L \tag{2.6}$$

The coefficient of the highest derivatives in Eqs. (2.5) and (2.6) is positive for any value of the angle β, since it is the determinant of the positive definite matrix of the kinetic energy E of the system (see relations (2.1) and (2.2)).

If the resistance to the rolling of the wheel is taken into account, assuming that this resistance depends on the velocity $\dot{x}$, then not only an angular acceleration $\dot{\omega}$ but, also, an angular velocity ω enters into Eqs. (2.4). The angular velocity of the wheel also enters into Eqs. (2.4) if a torque L is developed by the electrical drive and, the back EMF in the winding of the motor, which is proportional to $\dot{\beta} - \omega$, is taken into account. Under these conditions, it is not possible to "single out" an equation of the form (2.5), which alone describes the oscillations of the pendulum.

Introducing the dimensionless time τ and the dimensionless torque μ as in formulae (1.2), we transform Eqs. (2.5) and (2.6) to the form

$$(1-d^2\cos^2\beta)\beta'' + d^2\beta'^2\sin\beta\cos\beta - \sin\beta = (1+e^2\cos\beta)\mu \tag{2.7}$$

$$(1-d^2\cos^2\beta)\sigma' - e^2\beta'^2\sin\beta + e^2\sin\beta\cos\beta = -e^2\left(\frac{e^2}{d^2}+\cos\beta\right)\mu \tag{2.8}$$

Here

$$\sigma = \varphi', \quad d^2 = \frac{a_{12}^2}{a_{11}a_{12}} < 1, \quad e^2 = \frac{a_{12}}{a_{11}}$$

(σ is the dimensionless angular velocity of the wheel). The inequality $d^2 < 1$ can be proved directly, and its correctness also follows from the fact that the determinant of the positive definite kinetic energy matrix, being positive for all values of β, is also positive when $\beta = 0$. Note that system (2.7), (2.8) only contains two dimensionless parameters d and e.

Equation (2.7), which describes the oscillations of a pendulum with its suspension point at the centre of a wheel, contains the angle β with its own first two derivatives and does not contain the angular velocity ω of the wheel. At the same time, the inertial and geometrical characteristics of the wheel appear in the equation and affect the behaviour of the pendulum under any control $\mu(t)$. If the behaviour of the pendulum is of interest but the motion of the wheel is not of interest, Eq. (2.7) can be investigated independently of Eq. (2.8). Equation (2.7) has a more complex form than the known equation (1.3), which describes the oscillations of a pendulum with a fixed suspension point.

When $\mu = 0$, position (1.6) is an unstable equilibrium state not only of Eq. (1.3) but, also, of Eq. (2.7). We will now consider the problem of stabilizing this equilibrium state of the pendulum. More accurately speaking, we construct the corresponding domain of controllability for Eq. (2.7) (linearized) and the stabilizing control, for which the domain of attraction is identical to the controllability domain and thereby turns out to be the largest possible domain. We then compare this domain with controllability domain (1.11), which has been constructed above for a pendulum with a fixed suspension point. This is one of the basic aims of this paper.

Equation (2.7), linearized about the state (1.6), has the form

$$a^2\beta'' - \beta = c\mu; \quad a^2 = 1-d^2, \quad c = 1+e^2 > 1 \tag{2.9}$$

If we put $a = 1$ and $c = 1$ in differential equation (2.9), it takes the form of (1.8). When there is no control, that is, when $\mu = 0$, one eigenvalue of Eq. (2.9) is positive ($1/a$) and the other is negative ($-1/a$).

We now reduce the second-order equation (2.9) to a system of two first-order equations in the canonical variables

$$y' = \frac{y}{a} + \frac{c}{a}\mu; \quad y = \beta + a\beta' \tag{2.10}$$

$$z' = -\frac{z}{a} - \frac{c}{a}\mu; \quad z = \beta - a\beta' \tag{2.11}$$

Differential equation (2.10) describes the behaviour of the "unstable "variable y, which corresponds to the positive eigenvalue $1/a$, while Eq. (2.11) describes the behaviour of the "stable "variable z, which corresponds to the negative eigenvalue ($-1/a$). With $a = 1$ and $c = 1$ relations (1.9) and (1.10) are obtained from (2.10) and (2.11) respectively.

We denote by P the set of initial states, for each of which a control $\mu(t) \in W$ exists such that the solution of Eq. (2.9) (of system (2.10), (2.11)) with this control comes to the equilibrium (1.6). This controllability domain P is described[10] by the inequality (see Fig. 2)

$$|y| < c\mu_0 \quad (|\beta + a\beta'| < c\mu_0) \tag{2.12}$$

On returning in (2.12) to the initial parameters of the system, we obtain the inequality

$$\left|\beta + \sqrt{1 - \frac{m(Rb)^2}{r^2[M(R^2+\rho^2)+mR^2]}}\beta'\right| < \left[1 + \frac{mRb}{M(R^2+\rho^2)+mR^2}\right]\frac{L_0}{mgb} \tag{2.13}$$

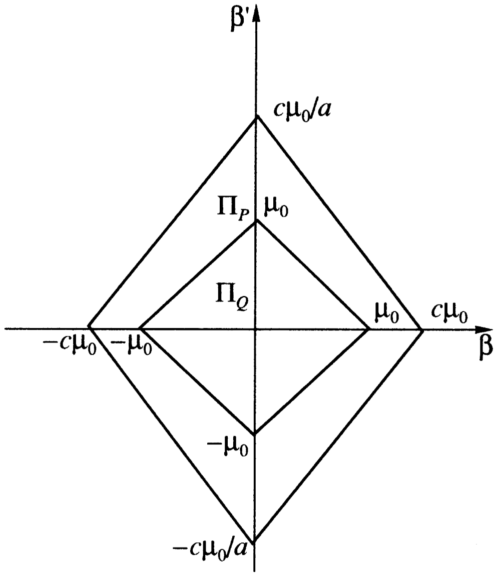

Fig. 4.

If the initial velocity $\beta'(0)=0$, then the constraint on the initial angle, at which the pendulum can be brought into the state of equilibrium (1.6), has the form

$$|\beta(0)| < c\mu_0 \tag{2.14}$$

If the initial angle $\beta(0)=0$, then the constraint on the initial velocity, at which the pendulum can be brought into the state of equilibrium (1.6), has the form

$$|\beta'(0)| < \frac{c}{a}\mu_0 \tag{2.15}$$

We will now compare the controllability domains Q and P, which have been constructed for a pendulum with a fixed and moving suspension point O respectively.

First, we note that the strip P (2.12) is wider than the strip Q (1.11) since $c>1$ and $a<1$ (see Fig. 2); it can thereby be said that it is larger, although the areas of the two domains are infinite. However, the domain Q does not wholly belong to the domain P, since the lines, bounding the domain Q, are inclined at a smaller angle to the abscissa axis than the lines, bounding the domain P. At the same time, it should not to be forgotten that the domains P and Q are constructed for the linear equations (1.8) and (2.9) and linearization is permissible is the values of the angle β and the angular velocity β' are sufficiently close to zero. However, if the values of β and (or) β' are sufficiently close to zero, the domain Q belongs to the domain P. We also note that the intervals (1.12) and (1.14) lie within the intervals (2.14) and (2.15) respectively, since $c>1$ and $a<1$. This means that the square Π_Q, the vertices of which are the ends of the intervals (1.12) and (1.14), lies wholly within the parallelogram Π_P, the vertices of which are the ends of the intervals (2.14) and (2.15) (see Figs. 2 and 4). Thus, the controllability domain of a pendulum with a fixed suspension point is smaller in a certain sense than the controllability domain of a pendulum with a moving suspension point; hence, a pendulum with a suspension point in a wheel is thereby more easily stabilizable. This is as would be expected since, in the second case, a torque L acts not only directly on the pendulum but it also gives an acceleration to the suspension point O, which promotes the stabilization of the pendulum.

As the mass of the wheel M increases, the value of a increases strictly monotonically and tends to unity when $M\to\infty$, and, as the mass M increases, the value of c decreases strictly monotonically and also tends to unity when $M\to\infty$. The lengths of the intervals (2.14) and (2.15), and therefore the dimensions of the parallelogram Π_P, decrease monotonically as the mass M increases, when $M\to\infty$, these intervals contract to the intervals (1.12) and (1.14) respectively and the parallelogram Π_P contracts to the square Π_Q. Hence, as the mass M increases, the domain (2.13) becomes smaller and, when $M\to\infty$, tends to the domain (1.11). This can be explained by the fact that, as the mass increases, the wheel becomes less mobile and hence the suspension point of the pendulum also becomes less mobile.

The radius of inertia of the wheel $\rho \le R$, and, therefore, the value of c, increases as the radius R decreases. Consequently, the range (2.14) of angles of deflection of the pendulum from the vertical, which are permissible from the point of view of its stabilizability, increases as the radius of the wheel decreases. This conclusion, which is based on a study of the linear model, is confirmed below using a non-linear model.

The effect of the different parameters of the system on the controllability domain can be estimated using formulae (2.13)–(2.15).

The linear feedback

$$\mu = \gamma y = \gamma(\beta + a\beta'), \quad \gamma < -1/c$$

and hence linear feedback with saturation

$$\mu = \begin{cases} -\mu_0 & \text{when} \quad \gamma(\beta + a\beta') \le -\mu_0 \\ \gamma(\beta + a\beta') & \text{when} \quad |\gamma(\beta + a\beta')| \le \mu_0 \\ \mu_0 & \text{when} \quad \gamma(\beta + a\beta') \ge \mu_0 \end{cases} \tag{2.16}$$

ensures the asymptotic stability of the solution (1.6) of Eq. (2.9) (of system (2.10), (2.11)). Under the feedback (2.16), the domain of attraction of the desired equilibrium (1.6) coincides[4–6] with the whole of the controllability domain P (2.12), that is, the feedback (2.16) realizes the maximum domain of attraction (in the linear approximation) and is optimal in this sense. The solution (1.6) of non-linear system (2.7), (2.16) will be asymptotically stable by virtue of Lyapunov's theorem on stability with respect to the first approximation.

Note that states which satisfy the conditions

$$\sin\beta = \pm(1 + e^2\cos\beta)\mu_0, \quad \beta' = 0 \tag{2.17}$$

will be equilibrium states of the non-linear equation (2.7) with the limiting control actions $\mu(t) = \mp\mu_0$. Each of Eqs. (2.17), which are non-linear with respect to the angle β, has a solution for a sufficiently small value of the quantity μ_0. They are distinguished by the sign and the larger in modulus of the quantity $\beta = \arcsin\,\mu_0$. (We recall that the expressions $\beta = \pm\arcsin\,\mu_0$ describe the angles of deflection from the vertical of a pendulum with a fixed suspension point in the case of the limiting control actions $\mu(t) = \mp\mu_0$.

Hence a pendulum with a suspension point in a wheel can be maintained in equilibrium at a larger angle of deflection from the vertical than a pendulum with a fixed suspension point. In fact, the torque applied to a pendulum which is deflected from the vertical, turns out not only to have a direct effect on the pendulum, tending to return it to the vertical position, but at the same time imparts an acceleration to the motion of the wheel (see Eqs. (2.7), (2.8))

$$\sigma' = \mp\frac{e^4}{d^2}\mu_0$$

which also tends to return the pendulum to this position.

If $R \to 0$, then $e \to \infty$ and the solutions of Eqs. (2.17) tend to $\pm\pi/2$. Consequently, as the radius of the wheel decreases, the value of the angle of deflection of the pendulum from the vertical, which is permissible from the point of view of its stabilizability, increases: this value tends to $\pi/2$ when $R \to 0$.

The state

$$\beta = 0, \quad \beta' = 0, \quad \sigma = 0 \tag{2.18}$$

is an equilibrium state of the system of Eqs. (2.7), (2.8) when $\mu = 0$. Eq. (2.8), linearized about the solution (2.18), has the form

$$a^2\sigma' + e^2\beta = -e^2\left(1 + \frac{e^2}{d^2}\right)\mu \tag{2.19}$$

When there is no control (when $\mu = 0$), the third-order system of equations (2.9), (2.19) has one positive eigenvalue $1/a$, one negative eigenvalue $(-1/a)$ and one zero eigenvalue. The controllability domain of this third-order system is described by the same inequality (2.12) as the controllability domain of the second-order equation (2.9). Inequality (2.12) does not contain the angular velocity σ, and, thereby, the possibility of bringing the system (2.9), (2.19) into the equilibrium (2.18) does not depend on the actual value of the velocity σ. Controllability domain (2.12) of system (2.9), (2.19) is a set of full measure in the three-dimensional phase space β, β', σ. Consequently, this system is fully controllable in the Kalman sense.[12]

The asymptotic stability of the state (1.6), but not the asymptotic stability of the state (2.18), holds under the control (2.16). At the same time, stabilization of the state (2.18) is possible since the system (2.9), (2.19) is controllable in a Kalman sense. The synthesis of the feedback, which stabilizes the state (2.18), does not give rise to any difficulties. It can be constructed, for example, by designating the eigenvalues of the closed system. This feedback must contain information on the actual value of the angular velocity σ.

This research was supported financially by the Russian Foundation for Basic Research (04-01-00105) and by the "State Support for Leading Scientific Schools" programme (1835.2003.1).

References

1. Stephenson A. On a new type of dynamical stability. *Mem and Proc Manchester Literary and Phil Soc* 1908;**52**(8 Pt 2):1–10.
2. Kapitsa PL. The dynamic stability of a pendulum with an oscillating suspension point. *Zh Eksper Teor Fiz* 1951;**21**(5):588–97.
3. Schaefer IF, Cannon RH. On the control of unstable mechanical systems, IFAC: 3rd Congr. London. 1966. Paper 601.
4. Grishin AA, Lenskii AV, Okhotsimskii DYe, Panin DA, Formal'skii AM. Synthesis of the control of an unstable object. *Izv Ross Akad Nauk Teoriya i Sistemy Upravleniya* 2002;**5**:14–24.
5. Beznos AV, Grishin AA, Lenskii AV, Okhotsimskii DYe, Formal'skii AM. A pendulum controlled using a flywheel. *Dokl Ross Akad Nauk* 2003;**392**(6):743–9.
6. Beznos AV, Grishin AA, Lenskii AV, Okhotsimskii DYe, Formal'skii AM. Control of a pendulum with a fixed suspension point using a flywheel. *Izv Ross Akad Nauk Teoriya i Sistemy Upravleniya* 2004;**1**:27–38.
7. Golubev YuF. A robot-balancer on a cylinder. *Prikl Mat Mekh* 2003;**67**(4):603–19.
8. Martynenko YuG, Formal'skii AM. Theory of the control of a monocycle. *Prikl Mat Mekh* 2005;**69**(4):569–83.
9. Martynenko YuG, Formal'skii AM. Control of the longitudinal motion of a single-wheel robot over an uneven surface. *Izv Ross Akad Nauk Teoriya i Sistemy Upravleniya* 2005;**4**:165–73.
10. Formal'skii AM. *The Controllability and Stability of Systems with Restricted Resources*. Moscow: Nauka; 1974.
11. Voronkov VS, Pozdeyev OD. Dynamics of the stabilization system of the magnetic suspension of the sensitive element of a gradiometer. *Izv Ross Akad Nauk, MTT* 1995;**1**:25–32.
12. Kalman RE, Falb PL, Arbib MA. *Topics in Mathematical System Theory*. New York: McGraw-Hill; 1969.

Translated by E.L.S.